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# Old and new issues in stellar evolution modelling

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**Abstract.** The knowledge of the evolutionary and structural properties of stars has achieved an high level of accuracy and maturity, thanks to an improved understanding of the physics at work in real stars. This notwithstanding, the current generation of stellar models is still affected by several - not always negligible - shortcomings related to our poor knowledge of some thermodynamical processes, nuclear reaction rates, as well as the efficiency of mixing/diffusive processes. These drawbacks have to be properly taken into account, when comparing theory with observations to derive evolutionary properties of both resolved and unresolved stellar populations. This paper reviews (some of) the major sources of uncertainty for the main evolutionary stages.

**Key words.** Stars: interiors – Stars: evolution – Stars: late-type

## 1. Introduction

The ability of the latest generation of stellar models to properly reproduce the observational properties of the various evolutionary sequences observed in the Colour-Magnitude diagram (CMD) of star clusters, is undoubtedly the crowning achievement of stellar evolutionary theory. Following this success, one is often tempted to use evolutionary results in an uncritical way, i.e., by taking these results at face value, without accounting for the associated uncertainties. Theoretical uncertainties do however still exist, as it is clearly shown by the differences amongst the results obtained by various research groups.

At the same time, the increasing amount of asteroseismological data for both field and cluster stars is opening a new window for studying with an un-precedented accuracy the reliability of the current generation of stellar models (see Chaplin & Miglio 2013, and references therein), besides providing also detailed information on the Galaxy formation and evolution (see e.g. Casagrande et al. 2014, 2016).

It is therefore important and timely to assess critically the accuracy and reliability of the theoretical stellar evolutionary framework. It is worth remembering that an early careful discussion of the uncertainties affecting stellar models for low-mass stars has been provided by Chaboyer (1995), who investigated the reliability of theoretical models for H-burning stars presently evolving in Galactic globular clusters (GGCs); the investigation has been extended to more advanced evolutionary stages by Cassisi et al. (1998, 1999), Castellani & degl'Innocenti (1999) and Gallart et al. (2005). A discussion of the drawbacks of stellar models for low- and intermediate-mass stars and

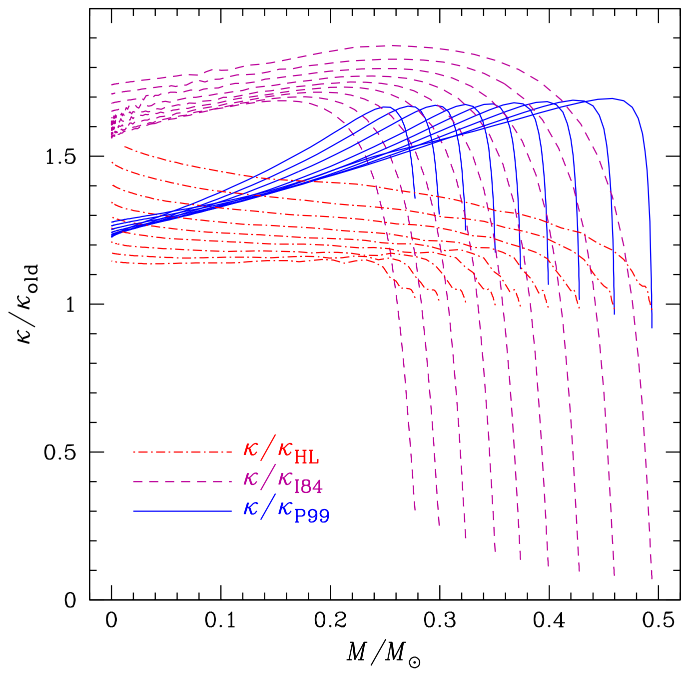


**Fig. 1.** Ratio of the conductive opacity provided by Cassisi et al. (2007) to previous evaluations by Hubbard & Lampe (1969, HL, dot-dashed lines), Itoh et al. (1984, I84, dashed lines), and Potekhin (1999, P99, solid lines). Each curve of a series corresponds to a specific RGB model for a $0.8M_{\odot}$ star, whose He-core mass is equal to the value of the mass attained at the end of the curve.

their impact on widely employed age, distance and chemical composition diagnostics has been provided by (Cassisi 2004, 2005, and references therein). A general overview on the state-of-the-art in the computation of stellar models for very low-mass (VLM) stars has been provided by Cassisi (2011). In these last years, several investigations as those performed by Valle et al. (2013, 2014) have addressed the problem of a quantitative and systematic evaluation of the cumulative propagation of physical uncertainties in stellar models by adopting a statistical approach.

Stellar evolution models are major tools in many astronomical research areas. Much fundamental information on resolved stellar populations otherwise inaccessible, as for example ages and the metallicities, is obtained by comparing observational data with theoretical predictions. Furthermore, evolutionary models play a crucial role also in studies of unresolved stellar populations, since they are a fundamental ingredient for the stellar population synthesis tools. On the other hand, the accuracy of the adopted evolutionary is crucial to derive robust insights about physical properties of galaxies when employing the techniques of stellar population synthesis.

## 2. The state-of-the-art in stellar modelling

An exhaustive discussion of the various shortcomings in stellar evolutionary computations is obviously outside the scope of this short review. We address here only (some of the) major problems still affecting model computations, such as the uncertainties in stellar input physics, the actual efficiency of mass loss during the Red Giant Branch (RGB) and the Horizontal Branch (HB) stages, the treatment of superadiabatic convection, the efficiency of mixing in the core of central H- and He-burning low-mass stars, and of atomic diffusion.

## 2.1. Input physics

In the last decade, the accuracy of theoretical models has been significantly improved thanks to updated and accurate predictions for both the thermal properties and the opacity of matter in the relevant regime for both interiors and atmospheres of low- and intermediate-mass stars. Concerning the equation of state (EOS), huge improvements have been made by the OPAL (Rogers & Nayfonov 2002) and FreeEOS[1] results. These EOSs take into account the most relevant non-ideal effects and allow an accurate description of the thermal properties of gas in both low- and intermediate-mass stars.

Another crucial ingredient in stellar modelling is the radiative Rosseland opacity; it is a fundamental property of the stellar matter that determines the amount of radiation absorbed and scattered in a stellar layer. Long-term intensive efforts to produce accurate opacity tables have been underway since several years, very important achievements being the results by the Opacity Project (Badnell et al. 2005, and references therein), the OPAL project (Iglesias & Rogers 1995, and references therein), and the more recent OPAS (Blancard et al. 2012) and Los Alamos opacity tables (Colgan et al. 2016). The need of more accurate opacity evaluations has become pressing in recent years due to the evidence that the revision in the solar chemical abundances (Asplund et al. (2005), but see also Asplund et al. (2009) and Caffau et al. (2011) for a significant revision of the solar metallicity) has worsened the good agreement that existed between standard solar models (SSMs) based on the old, (Grevesse & Sauval 1998) solar elemental abundances, and helioseismic constraints (see Serenelli et al. (2009) for a detailed description of this problem as well as for a thoughtful analysis of some tentative solutions). The renewed detailed scrutiny of the opacities used in SSM computations, has shown that the various existing Rosseland mean opacity computations are in very good agreement for various density and temperature conditions in the solar radiative zone. Some small differences do exist at the base of the solar outer convection zone, but in any case they are smaller than $\sim$ 5%.The relative contribution of individual elements to the Rosseland mean opacity have been also compared, showing this time large differences among the independent calculations. In any case, none of the most recent opacity tabulation is able to erase the current disagreement between SSM predictions and helioseismic results.

When the stellar matter is electron degenerate, heat transfer by electrons become an efficient –and in some case dominant– energy transport mechanism. In fact, when electron degeneracy is present, electrons cannot easily exchange momentum with other particles, because the quantum states with momentum lower than the Fermi value are occupied. Interactions become very rare and the electron mean free path increases hugely. When this occurs, a detailed evaluation of the electrin conduction opacity is crucial.

The physical conditions that require a detailed evaluation of the conductive opacity are encountered in the interiors of brown dwarfs, VLM stars with mass $M < 0.15M_\odot$, in the He-core of low-mass stars during their RGB evolution, in the CO core of Asymptotic Giant Branch (AGB) stars, as well as in white dwarfs (WDs) –in both core and a portion of the He-rich envelope – and envelopes of neutron stars.

Until the mid-1990s, the main sources of electron-conduction opacity tabulations suffered from a number of shortcomings that hugely hampered their accuracy. Several of these shortcomings have been addressed by Potekhin (1999) and Cassisi et al. (2007): these updated calculations can be used to compute opacities for arbitrary astrophysical mixtures at temperatures smaller than the 'Fermi temperature', and include the contribution from the electron-ion scattering and the electron- electron scattering in the regime of partial electron degeneracy. A comparison among the various sets of conductive opacities available in literature is shown in Fig. 1. The impact of the most

[1] `http://freeeos.sourceforge.net/`

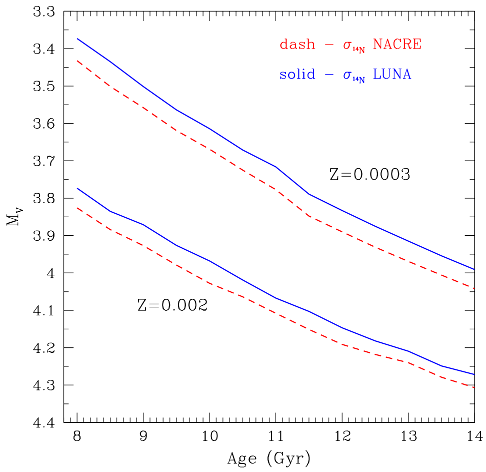


**Fig. 2.** The absolute V-band magnitude of the MS TO as a function of age for theoretical isochrones with Z = 0.0003 and Z = 0.002, as derived from stellar models computed with either the LUNA or the NACRE $^{14}$N(p, $\gamma$)$^{15}$O reaction rate.

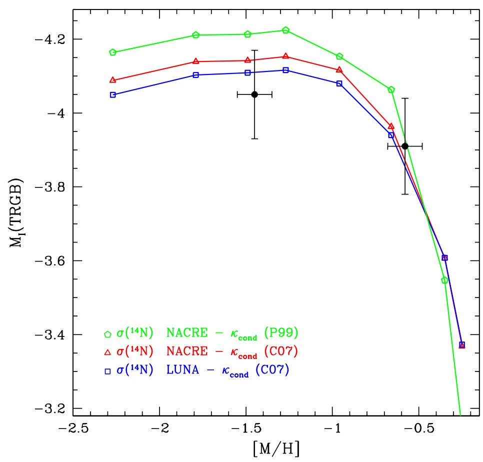


**Fig. 3.** The I-band absolute magnitude of the RGB tip as predicted by stellar models computed for various assumptions (see labels) about the $^{14}$N(p, $\gamma$)$^{15}$O nuclear reaction rate and the conductive opacity. The two points with error bars display the empirical estimates for the $\omega$ Cen and 47 Tuc.

updated conductive opacities on both RGB and HB models has been investigated by Cassisi et al. (2007).

Some important nuclear reaction rates are still affected by a significant uncertainty. For instance, one of the most important nuclear reaction in stellar astrophysics is the p-capture on $^{14}$N nuclei. At odds with the common idea that in low-mass stars the H-burning is always (fully) controlled by the nuclear reactions associated to the p-p chain, the nuclear reactions related to the CNO-cycle can be actually important. In fact, near the end of the Main Sequence (MS) stage, due to the paucity of H, the energy supplied by the H-burning becomes insufficient and the star reacts contracting its core in order to produce the requested energy via gravitation. As a consequence, both central temperature and density increase and, when the temperature reaches $\sim 15 \times 10^6 K$, the H-burning process is controlled by the CNO cycle, whose efficiency is critically dependent on the $^{14}$N(p, $\gamma$)$^{15}$O reaction rate, because this is the slowest reaction of the whole cycle.

In the past the rate for this reaction was uncertain, at least by a factor of 5, because all available laboratory measurements were performed at energies well above the range of interest for astrophysical purposes. The LUNA experiment (Formicola et al. 2003) has significantly improved the low energy measurements, obtaining an estimate which is about a factor of 2 lower than, for instance, the NACRE measurement. This new rate leads to a brighter and hotter MS turn-off (TO) for a fixed age (Imbriani et al. 2004; Weiss et al. 2005; Pietrinferni et al. 2010). The consequence is that, for a fixed MS TO brightness, the age - TO luminosity calibration predicts cluster ages, on average, $\approx$ 0.9 Gyr older than previous calibrations based on older estimates for this nuclear reaction rate, as shown in Fig. 2.

It is interesting to note that the change in the rate of this nuclear reaction has also a significant impact on the luminosity of low-mass stars at the RGB tip. Figure 3 shows the combined effect of variosu choices for the $^{14}$N(p, $\gamma$)$^{15}$O reaction rate and the conductive opacity, on the I-Cousins magnitude of the RGB tip of old stellar populations (one of the most important primary distance indicators -

see Salaris & Cassisi 1997, 1998, and Salaris et al. 2002) as a function of the metallicity.

Another nuclear reaction rate that would deserve a very accurate measurement is the one for the $^{23}$Na(p, $\alpha$)$^{20}$Ne reaction in the thermal regime suitable for the H-burning shell in intermediate-mass AGB stars. Indeed this nuclear reaction is not relevant for the energy budget, but it is quite important for an appropriate evaluation of the abundance of $^{23}$Na in the ejecta of these stars. This abundance plays an important role in the context of the multiple population phenomenon in GGCs: a reduction by a factor of $\sim$ 5 of this rate would enable to model better the O-Na anti-correlation disclosed by spectroscopic measurements see Renzini et al. (2015) and references therein for a review on the multiple population phenomenon in star clusters and related issues).

The $^{12}$C($\alpha, \gamma$)$^{16}$O reaction is one of the key reactions in stellar evolution. Its rate affects the C/O ratio in the stellar core at the end of the core He-burning stage and, as a consequence, WD cooling times. In addition, during the central He-burning phase, when the abundance of He in the convective core is significantly reduced, $\alpha-$captures on carbon nuclei becomes strongly competitive with the triple-$\alpha$ reactions in terms of contribution to the nuclear energy budget, with the consequence that the cross-section for this reaction has a strong influence on the core He-burning phase lifetime[2] (Cassisi et al. 2003). In massive stars this reaction rate affects all the subsequent hydrostatic burning stages and the nature of the remnant left behind after the core collapse.

It is known that this reaction has a resonance and a very low cross-section ($\sim 10^{-17}$ barn) at low energies, and the nuclear parameters are difficult to measure experimentally or to predict from theory. The accuracy of this reaction rate has been significantly improved thanks to Kunz et al. (2002) and by Hammer et al. (2005); however its estimate should be still affected by a $\approx 30\%$ uncertainty.

[2] The uncertainty on the $^{12}$C($\alpha, \gamma$)$^{16}$O reaction rate strongly affects the HB lifetime as $\Delta t_{HB}/t_{HB} \sim 0.10\Delta\sigma_{^{12}C}/\sigma_{^{12}C}$.

## 2.2. Diffusive processes

Besides convection, basic physics consideration require that additional transport processes are efficient within the stellar interior; they are driven by pressure, temperature and chemical abundance gradients, and by the effect of radiative pressure on the individual ions. These processes are often collectively called 'diffusive processes', and change slowly the abundances of chemical elements within radiative regions. In general, ions are forced to move under the influence of pressure as well as temperature gradients, that both tend to displace heavier elements towards the centre of the star, and of concentration gradients that oppose the above processes. The combination of these processes is commonly denoted as atomic diffusion. Radiation, that has a negligible effect in the Sun, pushes the ions towards the surface whenever the radiative acceleration on an individual ion species is larger than the local gravitational acceleration: this is the so-called radiative levitation. The speed of this diffusive flow depends on the collisions with the surrounding particles, as they share the acquired momentum in a random way. It is the extent of these collisional effects that determines the timescale of element diffusion within the stellar structure, once the physical and chemical profiles are specified. We refer to Burgers (1969) for an exhaustive description of the treatment of diffusion element transport in a multicomponent fluid.

Atomic diffusion has been neglected for many years in stellar model computations, for two reasons:i) It acts on very long timescales (more than $10^{13}$ yr are necessary for a particle to diffuse from centre to the surface of the Sun), and the effect on stellar evolution was considered negligible; ii) The inclusion of atomic diffusion in the calculations requires the computation of both spatial and temporal derivatives, that increase the complexity of stellar evolution codes. Nevertheless, observational constraints from the solar neutrino flux and helioseismology have forced the inclusion of atomic diffusion in SSM calculations (Cox et al. 1989), and in general, in low-mass stel-

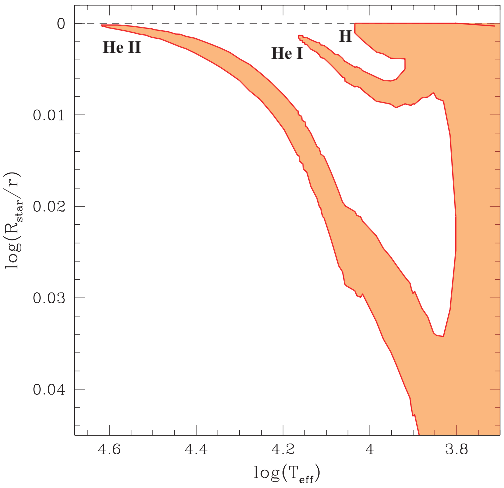


**Fig. 4.** The location of the convective layers associated to the HI, HeI and HeII ionization zones as a function of the $T_{eff}$ ZAHB location in canonical stellar models.

lar models (Cassisi et al. 1998; Michaud et al. 2010).

Following the Burgers formalism, a detailed solution of the atomic diffusion equations for a multicomponent fluid can be obtained, but to do this a preliminary estimate of the diffusion coefficients (entering in the diffusion equations) is mandatory.

Calculations of diffusion coefficients are based on the following assumptions: i) the gas particles have approximate Maxwellian velocity distributions; ii) the temperatures are the same for all particle species iii) the mean thermal velocities are much larger than the diffusion velocities, and iv) magnetic fields are unimportant. The evaluation of the diffusion coefficients requires the calculations of the interactions among particles under the assumption that collisions are dominated by the classical interaction between two point-charge particles, but more accurate estimates have also included quantum corrections (Schlattl & Salaris 2003). There is an intrinsic limitation related to the adoption of the Burgers formalism that causes an uncertainty in the diffusion coefficients of $\sim 10\%$, difficult to reduce further.

There is an additional and, perhaps, more important problem concerning atomic diffusion: although helioseismology strongly supports the efficiency of atomic diffusion in the Sun, spectroscopical measurements (as early shown by Castilho et al. 2000, Gratton et al. 2001, and Ramírez et al. 2001) of the iron content in cluster stars are in severe disagreement with the predictions provided by models with efficient diffusion. In fact, the iron abundance as measured at the surface of MS TO stars does not appear to be significantly reduced with respect the abundance estimated for RGB stars in the same cluster as one has to expect as a consequence of diffusion being at work. In addition, one needs to keep in mind that:

- radiative acceleration in the Sun can amount to about the 40% of gravitational acceleration (see e.g. Turcotte et al. 1998), and one can expect that its value is larger in more metal-poor, MS stars than in their metal-rich counterparts (the former being hotter than metal-richer objects). This means that radiative levitation (see below) could be actually important in GC stars;
- the occurrence of a slow mixing process below the solar convective envelope could help in explaining better the observed Be and Li abundances (Richard et al. 1996) and improve the agreement between the predicted sound speed profile and that derived from helioseismological data (Brun et al. 1999). It is quite common in stellar evolution analysis to refer to this non-canonical mixing process with the generic name of ‘turbulence’.

Regarding radiative levitation, when in the stellar layers the radiation field is strong and chemical elements are in a ionization state that favours interactions with photons, the transfer of momentum from photons to ions is very efficient and the acceleration imparted to the ion can become larger than the local acceleration of gravity, and can push chemical elements towards the surface. One can treat radiative levitation by considering in the diffusion equations an ‘effective gravity’, defined as $g_{eff} = g - g_{rad}$, with $g_{rad}$ being the acceleration attributed to a given ion species by the radiation field. The estimate of $g_{rad}$ extremely demanding in terms of computing time, because it requires an appropriate evaluation of the monochromatic total radiative opacity and of the monochromatic specific opacity associated to each chemical element. In addition this has to be repeated for all relevant chemical species, for each stellar layer and at each time step (for further details we refer to Cassisi & Salaris (2013); Michaud & Richer. (2008), and references therein). For all these reasons, only a few sets of stellar models accounting for radiative levitation have been produced so far.

A set of stellar models accounting simultaneously for atomic diffusion, radiative levitation and extra-mixing below the convective envelope of low-mass stars has been provided by Richard et al. (2002), VandenBerg et al. (2002) and Michaud et al. (2010). The main result is that with an ad-hoc calibration of the efficiency of turbulence, these models are able to reconcile helioseismology with the few available spectroscopical measurements of the iron abundance in at the TO of GGC stars. These models have been also very important for interpreting various observational features related to HB stars in star clusters[3], such as the 'jump' in the u Strömgren photometric passband (Grundahl et al. 1999), the so called low-gravity problem for hot-HB stars (Moehler 2001), and the chemical abundance anomalies in the outer layers of hot HB stars (Behr et al. 1999). These observations can be qualitatively explained by considering the effect of radiative levitation in HB stars with an effective temperature hotter than $\sim 11,500\ K$. In particular, the observation that photospheric abundances of some heavy element like iron, titanium and calcium are strongly enhanced with respect to their initial abundances, whereas other elements such as Mg and Si show very little – if any – enhancement, represents a direct proof of the efficiency of radiative levitation in the outer layers of HB stars. In fact, radiative levitation is expected to selectively enhance the surface abundance of heavy elements, according to the value of $g_{rad}$ for each individual ionic species.

[3] A short, but exhaustive, review of these issues can be found in Cassisi & Salaris (2013).

However, the same evolutionary computations show that, even when considering the outer convective zones associated to the HI, HeI and HeII ionisation zones (see Fig. 4) which contribute to limit the efficiency of the diffusive process, the predicted heavy elements enhancements are too large with respect spectroscopic measurements. It is tehrefore unavoidable to include also for HB models some ad hoc turbulent process operating in the outermost layers, with the aim of limiting the efficiency of the radiative levitation. The nature of the physical mechanism(s) that partially inhibit diffusive processes in HB (but also MS) stars is under debate. Turbulence needs to mix a fraction of the mass in the outer layers, to limit the efficiency of radiative levitation, and comparisons of models with observations of cluster HB stars and field sdB and sdO stars, dictate that this mixed mass fraction is of the order of $\sim 10^{-7}$ of the total stellar mass.

It is worth noting that the analysis of the rotational rate distribution along the HB of GGCs has revealed that hot HB stars and HB stars cooler than $T_{eff} \approx 11,500$ K have very different distribution of their projected rotational velocity ($v \sin i$). Stars cooler than this $T_{eff}$ limit can reach values of $v \sin i$ as high as $\sim 40$kms$^{-1}$, whereas hot HB stars show significantly lower rotational rates ($\sim 8$kms$^{-1}$). The presence of such bimodality in the rotational rate distribution is unclear, but it is striking the similarity of the $T_{eff}$ limit for the appearance of large surface abundance enhancements, and for the the discontinuity in the rotational rate. If this is not a coincidence, one is tempted to suggest that rotation and the related mixings (rotationally-induced meridional circulation and shear mixing) are the physical mechanism beyond the ad hoc turbulence required to be added to the model calculations (see Quievy et al. (2009), for a possible link between rotation rates and chemical anomalies). However, we need to remind that other processes such as mass loss (see below) may play a role.

### 2.3. Convection

One of the thorniest longstanding problems in stellar evolution theory is the treatment of convection. Despite the fact that stellar models are computed and critically compared with observational benchmarks since several decades, we are still employing a very approximate, and possibly not realistic, treatment of convection.

Indeed, we have to face with two distinct problems: i) How to treat the convection boundaries in the high density, largely adiabatic, inner layers, such as the convective core and the inner boundary of envelope convection zone; ii) how to calculate the efficiency of convection in the low-density, superadiabatic, outer layers.

### 2.3.1. Inner convection zones

During the central H- and He-burning stages, several physical mechanisms could extend the size of convective cores beyond the Schwarzschild boundary, such as the overshooting of convective elements in the stable surroundings, semi-convection, rotational mixing. These processes are still not satisfactorily understood by theory. Their combined effects are often modeled in the standard 1D calculations as a simple extension of the mixed convective core over a distance $d_{ov} = \lambda_{ov} \times H_P$, where $H_P$ is the pressure scale height at the edge of the convective core, and $\lambda_{ov}$ is a free parameter whose value has to be fixed. It is common to refer to $d_{ov}$ as the overshooting distance, even though overshooting may not be the only mechanism at work. This model is of course simplistic, but its simplicity reflects our current ignorance about the interface between convective and radiative regions. The value of the free parameter $\lambda_{ov}$ can be fixed by trying to match some empirical constraints such as the CMD morphology around the MS TO in young and intermediate-age star clusters and/or the properties of well studied eclipsing binary systems (see e.g. Pietrinferni et al. 2004).

When moving deeper inside the star, the pressure scale height steadily increases; this causes a large increase of the size of convective core in stars whose Schwarzschild convective boundary is fast shrinking, (e.g., for masses below $\sim 1.5M_\odot$) if the overshooting efficiency is kept fixed at a constant fraction of $H_P$. This occurrence imposes that the value of $\lambda_{ov}$ has to be decreased to zero for stars with small convective cores. Due to the lack of any physically-grounded prescription for how to deal with this issue, different assumptions about the trend of $\lambda_{OV}$ with the stellar mass can be adopted. The change of the isochrone morphology can be significant and different choices concerning the core overshoot efficiency in the critical mass range $1.1 \leq M/M_\odot \leq 1.5$ can mimic different isochrone ages. The obvious consequence is that the chosen trend of $\lambda_{OV}$ with mass, for masses with small convective cores, introduces an additional degree of freedom in stellar evolution models (Pietrinferni et al. 2004).

In these last years, a new approach has been envisaged to evaluate the extent of the convective core in central H-burning stars. This method is based on the measurement of the effects on oscillation modes caused by the sharp variations in the mean molecular weight profile at the boundary of the convective core. Preliminary investigations (Deheuvels et al. 2010; Silva Aguirre et al. 2013) have clearly shown that this is a promising avenue. However, only recently seismic surveys of large sample of stars are really exploiting the great potential of asteroseismology to provide tight constrains on the real extension of the convective core in MS stars (Deheuvels et al. 2015). As a consequence important improvements in this context are expected in the near future.

The treatment of convection at the boundary between the radiative layers and the fully convective core is still more complicated – and hence highly uncertain– during the central He-burning stage. For a detailed description of the sequence of events occurring in the inner core of low- and intermediate-mass stars during this nuclear burning stage we refer to Salaris & Cassisi (2005) and Cassisi & Salaris (2013), whilst here we only briefly mention the most relevant facts. Inside their convective cores, He-burning stars produce carbon via the triple$-\alpha$ reaction, and oxygen via $\alpha$ captures on carbon nuclei. Due to the larger radiative opacity of C/O rich mixtures with respect to He-rich ones, He-burning produces a growing discontinuity of the radiative gradient at the formal boundary of the convection zone if there is no convective overshoot to induce mixing beyond it. It has been shown (Castellani et al. 1971) that the mixing of a radiative shell surrounding the canonical convective core caused by an infinitesimally small overshoot beyond the Schwarzschild convective boundary, causes a local increase of the opacity and, in turn, a convective boundary instability. As a result, one expects a fast (compared to nuclear timescales) self-driving mechanism for the extension of the convective core: any radiative shell which is mixed as a consequence of the convective core overshooting, will definitely become part of the convective core.

The mixing of surrounding radiative shells produces a general decrease of the radiative gradient in the whole convective core, and the radiative gradient will eventually decrease to the value of the adiabatic gradient, after a number of radiative shells has been engulfed by the convective core. The layer where the radiative gradient becomes equal to the adiabatic one, marks the boundary of the resulting enlarged fully mixed convective core. This situation remains stable until the central He mass fraction decreases below a given value (typically ~0.7). At this point a new feature appears.

During the self-driving mass extension of the convective core $\nabla_{rad}$ develops a local minimum. Addition of more radiative shells surrounding the progressively larger convective core will decrease $\nabla_{rad}$ to the value of the adiabatic gradient at the location of the minimum of the radiative gradient. The main problem is then related to the treatment of the intermediate convective zone located between the minimum of $\nabla_{rad}$ (that now equals $\nabla_{ad}$) and the outer radiative zone. A full mixing between the convective core located inside the minimum and the external convective shell cannot occur, because at its minimum the radiative gradient is equal to the adiabatic one. If full mixing were to happen, the minimum of $\nabla_{rad}$ would decrease below the local value of $\nabla_{ad}$ and one would have the contradiction of a fully mixed region where $\nabla_{rad}$ is however locally smaller than $\nabla_{ad}$. As a consequence, the convective shell outside the minimum of $\nabla_{rad}$ can no longer be mixed with the inner core. A solution to this problem is the formation of an extended, partially mixed region – *semiconvective*– between the minimum of $\nabla_{rad}$ and the outer radiative zone. Inside the semiconvective region, the chemical composition is determined by the condition that $\nabla_{\rm rad} = \nabla_{\rm ad}$.

The uncertainty in the mixing treatment worsens as the core He-burning phase progresses. If no specific approach is used, the phenomenon of the so-called 'breathing pulses' occurs, that are a recursive phase of expansion of the convective core occurring when the He abundance in the core becomes quite low, and the feedback from the energy released by the $^{12}C(\alpha,\gamma)^{16}O$ nuclear reaction dominates with respect the triple$-\alpha$ reactions (due to the paucity of He nuclei in the core).

The occurrence of the breathing pulses in real stars has been severely challenged by observations (star counts), and their occurrence in stellar models is attributed to the approximation of instantaneous mixing in the convective zones adopted in the model computations. Different methodological approaches have been designed to inhibit the breathing pulses, but they lead to different predictions for the following evolutionary phases

The previous analysis of element mixing during the He-burning phase as discussed in the literature, has been performed without any proper hydrodynamical treatment, by simply considering the Schwarzschild criterion for convection (even in the semiconvective region where, due to the non-uniform He-profile, the Ledoux criterion would be in principle more suitable) using the predictions provided by the Mixing Length Theory (MLT) to determine the timescale of the convective boundary propagation, and assuming instantaneous mixing events. Alternatives to the scenario described before are to disregard the self driving mechanism and later on the onset of semiconvection (commonly denoted as Bare Schwarzschild model, or BSM scenario) or the inclusion of a substantial amount of overshooting beyond the Schwarzschild boundary (in case of semiconvection the self driving mechanism of the convective core boundary needs only an infinitesimally small amount of overshooting to be efficient). Indeed, a (relatively small) overshooting extension by $\sim 0.1 - 0.2H_P$ beyond the Schwarzschild boundary in the core produces core He-profiles, evolutionary lifetimes and evolutionary tracks almost coincident with the results from models including semiconvection.

So far, all investigations devoted to figure out what is the most appropriate mixing scheme during the central He burning phase have been based on the analysis of star counts in GGCs to infer the relative lifetime of core He-burning stars and Asymptotic Giant Branch stars, because - as mentioned - both of them are strongly dependent on mixing prescriptions (see, e.g. Cassisi et al. (1998), Constantino et al. (2015) and references therein). However, as discussed in the previous section, the central He-burning lifetime hugely depends on the poorly-determined rate of the $^{12}C(\alpha,\gamma)^{16}O$ nuclear reaction, and this occurrence strongly hampers the possibility of achieving a robust conclusion on this issue.

An alternative approach is offered by the analysis of the relative luminosity of the AGB clump with respect the HB in star clusters: this possibility is due to the evidence that the AGB clump luminosity is affected by both the treatment of the semiconvection layers and the technique for suppressing the breathing pulses (Cassisi & Salaris 2013).

The situation would improve if observational constraints specific to the internal structure of these stars would be available. Asteroseismology of thousands of RG stars observed by CoRoT and Kepler is nowadays changing the situation. We can now use the pulsation frequencies to place tight constraints not only on the fundamental stellar properties, but also to probe their internal structure (see, e.g. Chaplin & Miglio (2013) and references therein). In particular, Montalbán et al. (2013) have firstly shown that the frequencies of oscillation modes detected in core He-burning stars are sensitive diagnostics of the chemical and thermal stratification of the convective core, providing us with a novel and independent constraint, specifically for core structure of He-burning stars. First analyses have been recently performed by Bossini et al. (2015) and Constantino et al. (2016). The results appear still preliminary; in fact Bossini et al. (2015) suggest that a model with a moderate overshooting region in which an adiabatic thermal stratification is accounted for, should provide a better match with the asteroiseismic constraints as well as with the AGB clump luminosity; whereas Constantino et al. (2016) find that only models with a large convective core - as those accounting for a large overshoot efficiency - are able to reproduce the observed values of the asymptotic period spacing of the gravity oscillation modes. It is clear that we are still at an early stage in developing asteroseismic tools for constraining models of advanced evolutionary stages, and maybe we need to understand better how to manage the information coming out from the study of stellar oscillations as well as the possible selection effects in the asteroseismic population studies, as discussed by Constantino et al. (2016).

### 2.3.2. Superadiabatic convection

The temperature gradient throughout the bulk of the convective envelope of stellar models can be approximated by the adiabatic value. However, in the outer layers close to the stellar surface the gradient becomes strongly superadiabatic. To determine the local value of the gradient in these outer layers the MLT is almost universally used. The MLT contains four free parameters, whose values affect the predicted $T_{eff}$ of the stellar models. Three parameters are fixed a priori[4] (and define what we denote as the MLT flavour), and the only one left to be calibrated is $\alpha_{MLT}$, the ratio of the mixing length to the local pressure scale height $H_P$, that provides the scale length of the convective motions. As a general rule, an increase of $\alpha_{MLT}$ corresponds to an increase of the con-

[4] We refer to Salaris & Cassisi (2008) for a detailed analysis of the impact of choosing different values for these free parameters on stellar model computations. However, as a general rule, these authors have shown that sets of models –based on distinct MLT flavours but the same physical inputs– provide consistent results with only minor differences, once the mixing length has been properly calibrated on the Sun. The same sets of models are also able to reproduce the $T_{eff}$ of RGB stars in GGCs.

vective transport efficiency and an increase of the stellar model $T_{eff}$.

In stellar evolution calculations the value of $\alpha_{MLT}$ is usually calibrated by reproducing the radius of the Sun at the solar age with a SSM. This solar calibrated $\alpha_{MLT}$ is then kept fixed in all evolutionary calculations of stars of different masses and chemical compositions. The exact numerical value of $\alpha_{MLT}$ varies amongst calculations by different authors because variations of input physics and choices of the outer boundary conditions affect the predicted model radii and $T_{eff}$ values, hence require different $\alpha_{MLT}$ values to match the Sun. It is clear that, even if a physics input employed in the stellar model computations is not accurate, it is possible to mask this shortcoming –at least from the point of view of the predicted $T_{eff}$– by simply recalibrating $\alpha_{MLT}$ on the Sun. This guarantees that the models always predict correctly the $T_{eff}$ of at least solar-type stars. However, since the extension of the superadiabatic layers is larger in RGB stars, theoretical RGB models are much more sensitive to $\alpha_{MLT}$ than MS ones. Therefore it is not safe to assume a priori that the solar calibrated value of $\alpha_{MLT}$ is also adequate for RGB stars of various metallicities. A source of concern about an a priori assumption of a solar $\alpha_{MLT}$ for RGB computations comes from the fact that recent models from various authors, all using a suitably calibrated solar value of $\alpha_{MLT}$, do not show the same RGB temperatures. This means that –for a fixed empirical RGB temperature scale– the calibration of $\alpha_{MLT}$ based on RGB $T_{eff}$ estimates would not provide always the solar value. A comparison of independent sets of RGB stellar models (Salaris et al. 2002) computed with the same initial chemical composition and solar calibrated values of $\alpha_{MLT}$ shows that these models can predict a different $T_{eff}$ scale for the RGB: a realistic estimate of the current uncertainty on this $T_{eff}$ scale is of the order of $200 - 300$ K. The reason for this discrepancy must be due to some difference in the input physics which is not compensated by the solar calibration of $\alpha_{MLT}$ (VandenBerg et al. 2008). This occurrence clearly points out the fact that one cannot expect the same RGB $T_{eff}$ scale from solar calibrated models that do not employ exactly the same input physics.

The MLT formalism provides only a very simplified description of convection, and several attempts to introduce non-locality in the MLT have been made (see e.g. Deng et al. 2006, and references therein). These 'refinements' are often complex and introduce additional free parameters to be calibrated. The alternative model by Canuto & Mazzitelli (1991) and Canuto & Mazzitelli (1992) includes a spectrum of eddy sizes (rather than the one-sized convective cells of the MLT) and fixes the scale length of the convective motions to the distance to the closest convective boundary. Recently, Pasetto et al. (2014) have presented a novel non-local and time-dependent model based on the solution of the Navier-Stokes equations for an incompressible perfect fluid – the so called scale-free convection theory (SFC) –, that does not contain free parameter. A few stellar models based on this SFC have been recently presented by Pasetto et al. (2014): these preliminary computations seem to shown that, at least for MS stars, the SFC theory yields results very similar to those derived from a solar calibrated MLT regarding the extension of the convective zones, temperature gradients, and energy fluxes, whereas for RGB stars the differences between sets of models based on the two convection theories are significant. However, very recently the physical basis for this new approach for the treatment of the superadiabatic convection has been strongly questioned by Miller Bertolami et al. (2016).

An alternative approach to model the superadiabatic layers of convective envelopes is based on the computation of realistic multi-dimensional radiation hydrodynamics (RHD) simulations of atmospheres and convective envelopes –where convection emerges from first principles– that cover the range of effective temperatures, surface gravities, and chemical compositions typical of stars with convective outer regions. These simulations have reached nowadays a high level of sophistication (Nordlund et al. 2009) and for ease of implementation in stellar evolution codes, their results can be used to provide an 'effective

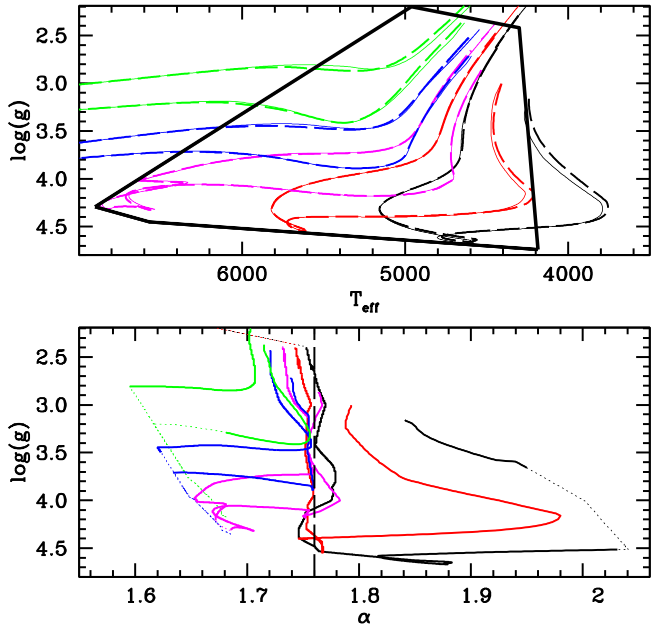


**Fig. 5.** Stellar evolution tracks in the $\log g - \log T_{eff}$ diagram for the following masses (from right to left): 0.7, 1.0, 1.4, 2.0, 3.0$M_{\odot}$. The region enclosed by the thick black boundary is the g-$T_{\rm eff}$ range covered by the RHD simulations. Thin solid lines denote fully consistent calculations with the RHD calibrated variable mixing length and T($\tau$) relationships. The lower panel shows the evolution of $\alpha_{\rm MLT}$ along each track. The dotted portion of each sequence denotes the region where the $\alpha_{\rm MLT}$ values have been extrapolated. Dashed lines in the upper panel display tracks calculated with a constant, solar-calibrated, $\alpha_{\rm MLT} = 1.76$, and the calibrated $T(\tau)$ relationship.

hydro-calibration' of $\alpha_{\rm MLT}$, even though RHD simulations do not confirm the basic MLT picture of columns of convective cells. In this context, Trampedach et al. (2013) produced a grid of convective atmosphere/envelope 3D RHD simulations for the solar chemical composition. The same grid of 3D RHD simulation have been matched (Trampedach et al. 2014a) to 1D hydrostatic equilibrium, spherically symmetric envelope models to calibrate $\alpha_{\rm MLT}$ as function of gravity and effective temperature. Interestingly, these RHD simulations provide a value for the $\alpha_{\rm MLT}$ for the Sun equal to $1.76 \pm 0.03$.

Moreover, the same RHD simulations have been employed by Trampedach et al. (2014b) to calculate g- and $T_{\rm eff}$-dependent temperature relations as a function of the $\tau$ Rosseland optical depth. The availability (Trampedach et al. 2014a) of a numerical routine to calculate a g- and $T_{\rm eff}$-dependent RHD-calibrated $\alpha_{\rm MLT}$ and T($\tau$) relations and Rosseland opacities consistent with the opacities used in the RHD simulations, enables stellar evolution calculations where boundary conditions, superadiabatic temperature gradient and opacities of the convective envelope are consistent with the RHD simulations[5]. Salaris & Cassisi (2015) have been the first to present stellar evolution calculations where this 3D RHD-calibration of $\alpha_{\rm MLT}$ is self-consistently included in a stel-

[5] We remark that it is particularly important to use both the RHD-calibrated $\alpha_{\rm MLT}$ and T($\tau$) relations, because the $T_{\rm eff}$ scale of the stellar models depends on both these inputs as discussed by Salaris et al. (2002) and VandenBerg et al. (2008) .

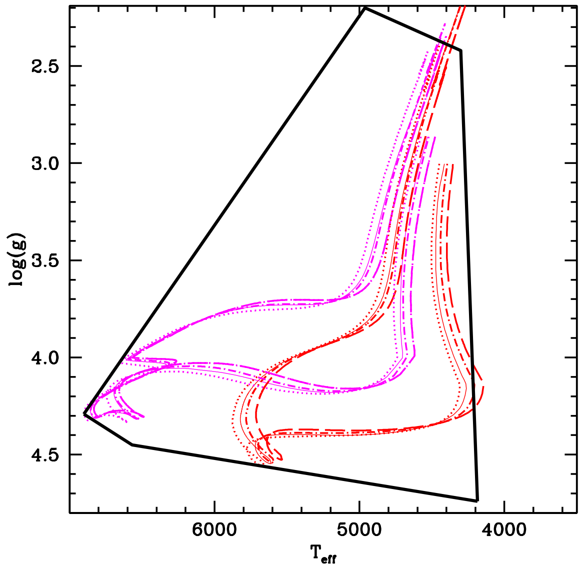


**Fig. 6.** As Fig. 5 but only for the 1.0 and 1.4$M_\odot$ stellar models and various assumptions about the T($\tau$) relation: thin solid lines denote fully consistent calculations with the RHD calibrated variable $\alpha_{\rm MLT}$ and T($\tau$) relationships; dotted, dash-dotted and dashed lines display tracks calculated with constant $\alpha_{\rm MLT}$ = 1.76 and the Eddington, Krishna Swamy (1966) and Vernazza et al. (1981) $T(\tau)$, respectively.

lar evolution code. Figure 5 shows some results from the quoted investigation: a comparison between fully consistent model calculations with the RHD calibrated variable $\alpha_{\rm MLT}$ and T($\tau$) relationships, and models based on a constant, solar calibrated $\alpha_{\rm MLT}$ = 1.76, and the same T($\tau$) relationship. From the point of view of the predicted $T_{\rm eff}$ scale, models calculated with constant RHD calibrated solar mixing length $\alpha_{\rm MLT}$ = 1.76 are very similar to, and often indistinguishable from, the models with variable $\alpha_{\rm MLT}$, maximum differences being at most $\sim$ 30 – 50 K. On the other hand, the same analysis has shown that the RHD-calibrated T($\tau$) relation is more relevant than the use of a variable $\alpha_{\rm MLT}$, in setting the effective temperature scale of the models as shown in Fig. 6.

## 3. Mass-loss in advanced evolutionary stages

Another major problem in stellar evolution theory is the treatment of the mass loss (ML) during the advanced evolutionary stages. The ML efficiency during the RGB stage controls the $T_{\rm eff}$ –hence the colour– of the model along the following HB phase. During the AGB the efficiency of ML determines the maximum luminosity and truncation of the AGB evolution, hence the contribution of the star to the integrated infrared flux of the stellar population, as well as its contribution to the chemical evolution of the interstellar medium.

Reliable empirical ML determinations, as well as a comprehensive physical description of the involved processes are still lacking. So far, there is a lack of any empirical law directly calibrated on Population II giants. Indeed, only a few, sparse estimates of ML for giants along the brightest portion of the RGB and AGB do exist. From a theoretical point of view, our knowledge of the ML timescales, driving mechanisms, dependence on stellar parameters and metallicity is also very poor. The consequence is that there is little theoretical or observational guidance on how to incorporate ML into stellar model computations.

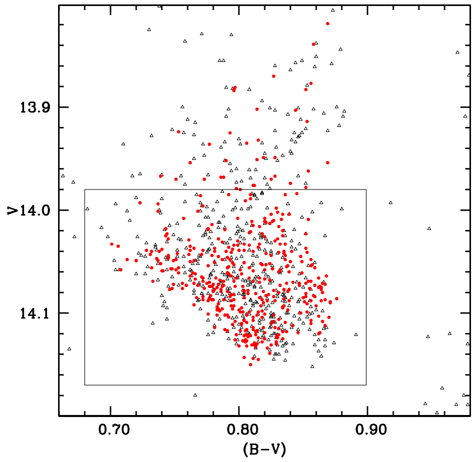


**Fig. 7.** A comparison between the observed HB (circles) in the GGC 47 Tuc, and a synthetic HB model obtained by assuming an uniform He distribution between Y=0.256 and 0.286, and an average –He-independent– mass lost by the RGB progenitors equal to $\Delta M \approx 0.23 M_{\odot}$, with a Gaussian spread by $0.005 M_{\odot}$.

As for the ML efficiency along the RGB, mass loss rates are customarily parametrized in stellar evolution calculations by means of simple relations like the Reimers formula (Reimers 1975). A few other formulae, which are variants of the Reimers one, have been proposed in more recent years (Catelan 2009) but there is no a priori reason for choosing one amongst the different alternatives. These prescriptions are essentially scaling relations between mass loss rates and global stellar parameters like, e.g., surface bolometric luminosity and gravity, effective temperature and/or radius. The zero point of these scaling relations is typically set by a free parameter ($\eta$) that needs to be calibrated.

The last decade has witnessed a growing amount of empirical data concerning ML estimates for Pop. II red giants, based on a more direct approach as the detection of outflow motions in the outer regions of the atmospheres (see e.g. Vieytes et al. 2011, and reference therein), or the detection of the circumstellar envelopes at larger distances from the stars (see i.e. Origlia et al. (2007, 2010), Momany et al. (2012) and references therein). These empirical analyses suggest that the 'actual' ML law should be significantly different (flatter) than the Reimers formula, that seems to be ruled out at the $3\sigma$ level. In addition, the RGB ML phenomenon seems to be not a continuous process but an episodic phenomenon, whose efficiency does not appear to be strongly correlated with the metallicity.

Very recently, Heyl et al. (2015a,b) applied an alternative approach to estimate the mass lost by RGB stars in the Galactic GC 47 Tuc. These authors determined the rate of diffusion of stars through the cluster core, using a sample of young white dwarfs, and compared the radial distribution of upper MS, RGB and HB stars. As a result, they found that the radial distributions of the various classes of stars are nearly identical, showing that there has been very little time for the young WDs to have diffused through the cluster since their progenitors lost mass. Based on these dynamical consideration, Heyl et al. (2015b) estimated that up to two-thirds of the $\sim 0.4 M_{\odot}$ that 47 Tuc stars are expected to lose between the end of the MS and the beginning of the WD sequence, is shed shortly before the start of the WD cooling, and a typical HB stellar mass of the order of $\sim 0.65 M_{\odot}$ (that corresponds to a mass loss of about $0.25 M_{\odot}$ along the RGB) is excluded by comparing the radial distribution of HB stars and MS stars.

However, the need for a significant mass loss during the RGB is necessary in order to reproduce the observed cluster HB morphology, as shown by Salaris et al. (2016). The main result of their analysis is shown in Fig. 7. The comparison between synthetic HB models and data shows that the observed morphology can be reproduced only by accounting for an average mass lost during the RGB equal to $\Delta M \approx 0.23 M_{\odot}$ and an almost negligible Gaussian spread of $0.005 M_{\odot}$ around this value. In addition one needs to include a suitable spread in the initial helium abundance of the cluster stars (as constrained from other features in the CMD - see Milone et al. (2012) for details), as expected from the presence of multiple populations within individual GGCs (see

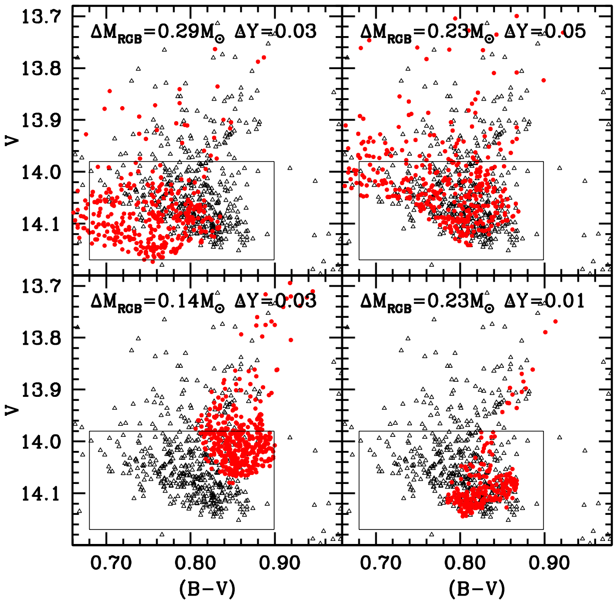


**Fig. 8.** As Fig. 7, but for various assumptions about the spread in the initial He abundance or the average value of the mass lost during the RGB stage.

Piotto et al. 2015, and references therein), that display large photospheric abundance variations of elements involved in high-temperature CNO-cycle H-burning (like O-Na, C-N and Mg-Al anticorrelations).

Figure 8 shows how the shape and location of the synthetic HB are affected by a change in the assumed average efficiency of the RGB mass loss or the spread in the initial He abundance. Variations of $\Delta M_{RGB}$ keeping the reference He distribution fixed, and considering the possible sources of uncertainty related to the cluster properties such as reddening, chemical composition and age, and the additional empirical constraint coming from the location of the RGB in the CMD, Salaris et al. (2016) found that the average mass lost during the RGB ranges from a minimum value of $\sim 0.17 M_\odot$ (for a GC age of 12.5 Gyr) to a maximum value of $\sim 0.21 M_\odot$ (for a GC age of 10.5 Gyr). This analysis suggests a discrepancy between the information coming from cluster dynamics and CMD modelling of the HB. A comparison between the results from these two techniques applied to other clusters is required, to gain more insights about the origin of this apparent disagreement.

As for the efficiency of mass loss during the HB evolution, this issue is important due to the already mentioned discovery of peculiar chemical patterns at the surface of hot HB stars. In fact, ML could act as a mechanism able to limit the effect of radiative levitation. As a consequence, the presence of these chemical peculiarities poses important constraints on the ML efficiency in HB stars: the observed chemical patterns can only be explained if mass-loss rates are in the range $(10^{-14} - 10^{-12}) M_\odot/\mathrm{yr}$. To date, the only theoretical analysis (Vink & Cassisi 2002) of the efficiency of HB mass-loss in the hypothesis that it is driven by radiation pressure on spectral lines, has derived rates $\sim 10^{-12} M_\odot/\mathrm{yr}$ at solar metallicity, that should decrease by one or two orders of magnitude at metallicities typical of GGCs.

Recently, the apparent lack of second generation (Na-rich) stars along the AGB and the relatively low value of the parameter $R_2$

(the ratio between AGB and HB stars) in the GGC NGC 6752 has been interpreted by Campbell et al. (2013) as an evidence of the fact that all NGC 6752 HB stars hotter than $T_{eff} \approx 11,500$ K fail to reach the AGB due to enhanced mass loss during the HB stage, possibly associated with the surface metal enhancement caused by radiative levitation. By employing the Reimers mass loss law, Campbell et al. (2013) estimated a value for the free parameter $\eta \approx 10$ to explain the lack of Na-rich stars along the AGB, an enhancement of a factor $\sim 20$ compared to the value typically used in the calculation of RGB models. This issue has been critically investigated by Cassisi et al. (2014). They found that the mass-loss rates required to force the HB progeny to miss the AGB stage are of the order of $10^{-9} M_{\odot}/yr$ , hence significantly higher than current, previously mentioned, theoretical and empirical constraints. By making use of synthetic horizontal branch simulations, Cassisi et al. (2014) have demonstrated that canonical stellar models predict correctly the $R_2$ ratio, without the need to invoke a lack of asymptotic giant branch stars due to mass loss. The same simulations predict however the presence of (a few) Na-rich stars along the cluster AGB. Interestingly enough, quite recently García-Hernández et al. (2015) have provided sound empirical evidence actually supporting the presence of Na-rich stars among the AGB population of metal-poor Galactic GCs. All these results suggest that there is no need to account for a highly efficient mass loss process during the core He-burning stage.

The efficiency of mass loss during the AGB stage has a dramatic impact on both lifetimes and chemical yields. Although for AGB stars the mechanism responsible for the observed large mass-loss rates has been identified as the interaction between radiation pressure and dust particles, we still lack a physically grounded description of how mass loss mechanism operates in AGB stars. For instance, it is commonly believed that dynamics, playing a major role in the dust formation process, could be an essential ingredient in determining the AGB mass loss rates. Shock waves as those generated by stellar pulsation and/or convection processes, propagate outwards through the atmosphere and lift gas above the stellar surface, intermittently creating dense, cool layers where dust particles may efficiently form. The complicated, and still largely not understood, link among mass loss efficiency, pulsations and mixing processes makes difficult to derive an accurate mass loss recipe for AGB model computation (see van Loon 2008, and references therein).

The situation worsens when accounting for the fact that, in those AGB stars experiencing the third dredge-up, the chemical composition of the envelope and atmosphere changes with time, becoming enriched in carbon. To date, accurate and detailed radiation-hydrodynamics models accounting for both dust production and a variable C/O ratio (and metallicity) are not available. For such a reason, all AGB stellar model computations adopt theoretical or semi-empirical relations with some free parameter whose value is fixed by matching some empirical constraints.

## 4. Final remarks

Huge improvements have been made regarding the major physics ingredients entering in stellar models 'cooking' such as opacities, nuclear reaction rates, equation of state. Although there are still some issues –as the uncertain rates of some specific nuclear reactions– a very realistic description of the main input physics needed for stellar computations seems within reach. The same conclusion applies to the treatment of diffusive processes. On the other hand, we still face long-standing problems related to the treatment of convection in both stellar interiors and atmosphere. Despite the large efforts devoted to addressing this problem, we still need to use approximate numerical approaches with some (tunable) free parameter(s). In this context, important advances towards development of a comprehensive stellar convection theory may come from constraints obtained by current (and future) asteroseismic surveys.

As for the efficiency of mass loss along both the RGB and HB stages, we still miss a self-consistent, physically grounded, theory describing how it occurs and allowing to predict a priori the dependence of its efficiency on the stellar properties. Concerning the AGB stage, despite the huge complexity of the physical processes that regulate the mass loss efficiency, a realistic description of dust-driven mass loss in AGB stars could be achieved in the next decades. Needless to say that a better understanding of how to treat the efficiency of mixing processes is extremely important also in this context.

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